Sample ECE 174 Midterm Questions

1. VOCABULARY AND DEFINITIONS. *Define* the following terms.

   Vector Space; Linear Independence; Dimension; Norm; Triangle Inequality; Banach Space; Inner Product; Hilbert Space; Cauchy-Schwartz Inequality; Generalized Pythagorean Theorem; Projection Theorem/Orthogonality Principle; Adjoint Operator; Independent Subspaces; Complementary Subspaces; Orthogonal Complement; Projection Operator; Orthogonal Projection; Linear Inverse Problem; Ill–posed Linear Inverse Problem; Least–Squares Solution; Minimum Norm Least–Squares Solution; Moore–Penrose Pseudoinverse.

2. GEOMETRY OF LEAST SQUARES AND THE PROJECTION THEOREM. Consider the system $Ax = b$, $A \in \mathbb{C}^{m \times n}$. View $A : \mathbb{C}^n \to \mathbb{C}^m$ as a linear operator between two finite dimensional Hilbert spaces (of dimension n and m) over the field of complex numbers C.

   (a) What is the geometry induced on the domain and codomain of A by A? State in terms of $\mathbb{C}^n$, $\mathbb{C}^m$ and the "Fundamental Subspaces" of A and its adjoint. Give the dimensions of the subspaces and their geometric relationships to each other and the domain and codomain.

   (b) (i) Give a condition for the system $Ax = b$ to have a solution for *every* $b \in \mathbb{C}^m$. (ii) Give a condition for the system to have a *unique* solution, when one exists. (iii) When *neither* of these conditions holds, describe the solution possibilities in terms of b.

   (c) Assume only that $\text{rank}(A) = n$. (i) Characterize the optimal solution to the Least Squares Problem, $\min \|Ax - b\|^2$. (I.e., what *geometric condition* must the optimal solution satisfy?) (ii) Derive the Normal Equations (do *not* take derivatives). (iii) Does a unique optimal solution exist and why?

   (d) Now assume only that $\text{rank}(A) = m$. (i) Characterize the optimal solution to the Minimum Norm Problem, $\min \|x\|^2$ subject to $Ax = b$. (I.e., what *geometric condition* must the optimal solution satisfy?) (ii) Derive an explicit form of the optimal solution in terms of A and b (do *not* take derivatives).

   (e) (i) Give an exact expression (in terms of A and its adjoint) for the Moore-Penrose pseudoinverse, A^+, of A when $\text{rank}(A) = n$ (ii) Repeat for when $\text{rank}(A) = m$. (iii) Finally, show that when A is square ($m = n$) both of these expressions reduce to $A^+ = A^{-1}$.

3. OPERATOR ADJOINTS AND QUADRATIC OPTIMIZATION.

   (a) Solve the Weighted Least Squares Problem,

   $$\min_x \frac{1}{2}\|Ax - b\|_W^2 \,,$$

   where $A \in \mathsf{C}^{m \times n}$, $W = W^H > 0$, and $\text{rank}(A) = n$. Give the final solution explicitly in terms of W, A, and b *only* (or their hermitian transposes), using the appropriate inverses. Do *not* take derivatives or factor W. (You can assume that the standard 2-norm holds on the domain.)

   (b) Solve the Minimum Norm Problem,

   $$\min_x \frac{1}{2}\|x\|_\Omega^2 \quad \text{subject to} \quad Ax = b \,,$$

   where $A \in \mathsf{C}^{m \times n}$, $\text{rank}(A) = m$, and $\Omega = \Omega^H > 0$. Give the final solution explicitly in terms of Ω, A, and b *only* (or their hermitian transposes), using the appropriate inverses. Do *not* take derivatives or factor Ω. (You can assume that the standard 2-norm holds on the codomain.)

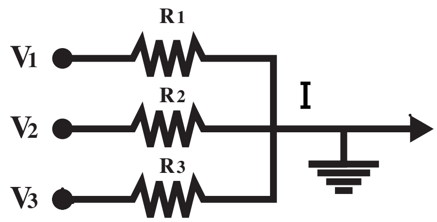

Figure 1: The three resistor values are given and fixed, as is the desired target current I. You are to determine the voltages V_1, V_2, and V_3 which will attained the target current while minimizing the power dissipated in the circuit. You must also determine the optimum (minimum) value of the power dissipated in the circuit.

4. SIMPLE APPLICATIONS. Do *not* use derivatives.

   (a) In the plane, R^2, suppose that repeated noisy measurements (say m of them) are made of a line through the *origin*. *Derive the least squares estimate of the slope* of the line based on your measured data.

   (b) Consider the three-resistor circuit shown in Figure 1 where I is a specified nonzero constant current and $R_1 \neq R_2 \neq R_3 \neq R_1$. ($i$) Find the voltages V_1, V_2, V_3 which will minimize the power dissipated in the resistors. (ii) Derive the optimum (minimum) power dissipated when the optimal voltages are used. (iii) Now let $R = R_1 = R_2 = R_3$ and show that the optimum power dissipated is $\frac{1}{3}$ the power dissipated when the simple solution corresponding to $V_2 = V_3 = 0$ is used.

   (c) In the plane, R^2, *derive the minimum distance from the origin to the line* $y = ax + b$. (The scalars a and b are both assumed to be nonzero.) Give the answer in terms of a and b.

   (d) (i) Determine the Normal Equations and the form of the pseudoinverse solution appropriate for determining a least–squares empirical fit to the forward I–V characteristic of a diode using the model,

   $$I = \alpha + \beta V^3 \,,$$

   and abstract data (V_k, I_k), $k = 1, \cdots, m$. (ii) Apply your solution to the specific numerical data,

V (mV)	0.00	0.50	1.00
I (mA)	-0.09	0.98	8.71